# Robust Optimal Investment-Reinsurance Strategies for a Jump-Diffusion Risk Model with Mean-reverting Stock Return

**Wu Yungao, Professor**

*Wu Yungao, Professor in Mathmatic, School of Economics and Management, North China Electric Power University, Beijing, 102206, China; Department of mathematics and Computer Engineering, Ordos Institute of Technology, Ordos, 017000, China*

**Objectives: This paper proposes a strategy of robust optimal investment reinsurance for insurance companies. It was assumed that the surplus procedure of the insurance company satisfies the jump-diffusion procedure. Insurance companies could invest their surplus funds in the financial market consisted of both risk assets and one risk-free asset. The price procedure of risk assets satisfies the stochastic procedure with a mean reversion rate. Considering the uncertainty of the model, the ambiguity-averse insurance firm aims to enhance the exponential utility of insurance surplus at terminal time. This paper has investigated the problem of robust optimal investment reinsurance and obtained the differential equation supported by the value function.**

**Key words: robust optimal strategies; proportion reinsurance; jump-diffusion; mean reversion process**
**Tob Regul Sci.™ 2021;7(6): 6100-6114**
**DOI: doi.org/10.18001/TRS.7.6.88**

## INTRODUCTION

The insurance company is an enterprise that deals with risks. To spread the risk of insurance, insurance companies usually sell insurance to reinsurance companies buy reinsurance to reduce risk. On the other hand, to make full use of the surpluses, the insurance company should pay more attention to it in the financial market, companies obtain excess returns through venture capital. Generally speaking, the reinsurance forms of the insurance firm have both non-proportional and proportional reinsurance, and insurance surplus could be invested in the risk-free assets of the banks such as bonds or deposits, also could be invested in risky assets such as stocks.

The reinsurance and optimal investment of insurance companies have been widely studied. Based on various objectives, a large number of research scholars have researched the optimal reinsurance and investment strategy[1-4]. However, the above literature assumes that the parameters in the insurance surplus procedure and the price procedure of financial assets can be accurately estimated. However, the parameters of the model could not be easily and accurately estimated, especially for the expected return rate (ERR) in the process of the risk asset price, which always has an inevitable estimation error. Due to the uncertainty of parameter estimation, the obtained investment and reinsurance strategies are not robust. This brings difficulties to investment decision-making. Thus, the ambiguity-averse insurance firm aims to devise a robust strategy. The main idea of a deal with model uncertainty is that the insurer finds the optimal strategy of investment under the worst-case probability measure. For example, Maenhout (2004) investigates the influence of ambiguity aversion on the usual dynamic investment-consumption strategy[5]. Maenhout (2006) describes the ERR of risk carrying assets by using a mean-reverting process and explores the influence of ambiguity aversion on the problem of dynamic investment-consumption[6]. Gu et al. (2018) assume that there is mispricing in the financial markets, and they solved the problems of reinsurance strategy and required robust optimal investment[7]. Guan & Liang (2019) investigate the robust optimal investment problem and the strategy of reinsurance from the perspective of multiple risky assets in the market[8], Zhang and Chen (2019) also investigated the same investment strategies from the context of jumps and defaultable risks[9].

In addition to the above studies, many works have been done to develop reinsurance and optimal investment strategies for insurance companies. Lin & Yang (2011) also examined these strategies for the diffusion risk model for maximizing terminal wealth.[10] The delayed reinsurance and optimal investment strategy of insurance companies with mean-variance is considered by Shen & Zeng (2014).[11] Li et al. (2015) study the insurance problem where the underlying claim process follows Brownian motion with drift assuming that the insurer purchases proportional reinsurance.[12] Sun et al. (2017) studied such strategies of insurance companies under ambiguity and the default risk model.[13] The issue of robust optimal investment-reinsurance is studied by Zhang et al. (2019) in the environment of model uncertainty.[14] Sun et al. (2019) considered the multiple dependent risks to shed some light on the problem of investment reinsurance for ambiguity-averse insurance firms.[15] While, Chen & Yang (2020) discussed the matter of robust optimal reinsurance investment with fuzzy price and relevant claims.[16] There are many studies on investment and reinsurance problems.[17-22]

This paper has supposed that the surplus procedure of the insurance firm satisfies the procedure of jump-diffusion. The insurers invest their surplus in risk-free assets and a risk asset whose ERR is consistent with the mean-reversion process. In the process of building the mathematical model, the main objective is enhancing the exponential utility of the expected wealth at the terminal time. Considering the uncertainty of the model parameters, ambiguity aversion investors want to seek a robust reinsurance and asset allocation strategy. We have been able to derivative the robust HJB equation that is satisfied by the value function through employing the control approach on the basis of the robust stochastic. Afterward, three ordinary differential equations are obtained based on the form of the value function.

## PROBLEM IDENTIFICATION FOR THE MODEL DYNAMICS

Suppose $(\Omega, \mathrm{F}, \mathbb{F}, \mathrm{P})$ be a filtered space of probability with filtration $\mathbb{F} = \{\mathrm{F}_t\}_{t\in[0,T]}$. $\mathbb{F} = \{\mathrm{F}_t\}_{t\in[0,T]}$ is created through three one-dimensional standard Brownian motions $\{M_0(t)\}$, $\{M_1(t)\}$ and $\{M_2(t)\}$, which are independent of each other. $T > 0$ represents the investment time horizon, which is a finite constant. We consider that $\mathbb{F}$ demonstrates right-continuous and $\mathrm{P}$-complete, and the $\mathrm{F}_t$ represents the data that is accessible until time $t$. Besides, it was also supposed that in the financial market there are no taxes or transaction costs, and trading could occur on a continuous basis.

In the following, we first introduce the insurer's jump-diffusion surplus process. The definition is shown as follows.

$$\mathrm{d}X(t) = c\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N(t)} Y_i + \sigma_0 \mathrm{d}M_0(t), \tag{1}$$

In Eq. (1), $c > 0$ shows the constant premium rate per unit of time. $\{N(t)\}_{t\in[0,T]}$ represents a Poisson procedure, and $\lambda > 0$ is a constant intensity. $\{Y_i, i \ge 1\}$ is a series of non-negative random variables with independent and identical distribution $F(y)$, and the corresponding expectations are $\mu = \mathrm{E}[Y_i]$, $\mu_2 = \mathrm{E}[Y_i^2]$. The moment-generating function $Y_i$ is indicated as $P_Y(t) = \mathrm{E}[e^{tY_i}]$. Constant is denoted by $\sigma_0 \ge 0$. $\{M_0(t)\}_{t\in[0,T]}$ implies one dimensional standard Brownian motion, which captures an extra source of uncertainty of the surplus process, such as

\

the uncertainty of premium incomes or aggregate claims.

Now we assume that the risk exposure is offset by the insurance firm by employing the reinsurance technique. We allow the insurer to render the proportional reinsurance with retention level $q(t)\in[0,1]$, in other words, when a claim $Y_i$ happened at $t$ the time, the insurance firm pays $q(t)Y_i$, and the reinsurer pays $(1-q(t))Y_i$. For the insurer, the expectation premium principle is adopted, i.e., $c=(1+\eta_0)\lambda\mu$, here $\eta_0$ represents the safety loading of the insurance firm. Furthermore, this paper supposes that the variance premium is adopted by the reinsurer, thus the reinsurer premium $\delta(q(t))=(1-q(t))\lambda\mu+\eta_1(1-q(t))^2\lambda\mu_2$, where $\eta_1$ is safety loading of the reinsurer. Yet in general, it was supposed that reinsurance is more expensive. This means the full reinsurance (all the claim is paid by reinsurance) is more expensive than the first one. Otherwise, the first insurer can reinsure the entire portfolio and gain profit without undertaking any claim. Thus, $\eta_1>(c-\lambda\mu)/(\lambda\mu_2)$.

In addition, to spread the business risk of potential insurance, we suppose that the insurance firm invests all the surplus into both risk-free assets and risky assets. The price process of risk-free asset satisfies

$$\mathrm{d}C(t)=rC(t)\mathrm{d}t, \tag{2}$$

$r>0$ stands for the interest rate of a risk-free asset. And the price procedure of the risky asset evolves according to the below equation:

$$\mathrm{d}S(t)=S(t)[\mu(t)\mathrm{d}t+\sigma\mathrm{d}M_1(t)], \tag{3}$$

where $\mu(t)$ is the appreciate rate and $\sigma>0$ represents the stock volatility. Moreover, the risk price in the market $\{\theta(t)=\frac{\mu(t)-r}{\sigma}\}_{t\in[0,T]}$ satisfies a mean-reverting Ornstein-Uhlenbeck procedure as follows:

$$\mathrm{d}\theta(t)=k(\bar{\theta}-\theta(t))\mathrm{d}t+\sigma_\theta[\rho\mathrm{d}M_1(t)+\sqrt{1-\rho^2}\mathrm{d}M_2(t)] \tag{4}$$

In Eq. (4), $k>0$ represents the extent of mean reversion, $\bar{\theta}>0$ represents the long-term average of the excess stock return, $\sigma_\theta>0$ denotes the volatility. $\{M_1(t)\}$ and $\{M_2(t)\}$ are independent. $\alpha(t)$ is the surplus amount that is channelized towards the risky asset at the time $t$. Denote $X^\pi(t)$ as the wealth procedure associate with strategy $\pi(t):=\{\alpha(t),q(t)\}_{t\in[0,T]}$. Subsequently, the wealth procedure of the insurance company with proportional investment and reinsurance fulfills the Eq. (5):

$$\begin{aligned}\mathrm{d}X^\pi(t)&=\alpha(t)\frac{\mathrm{d}S(t)}{S(t)}+(X^\pi(t)-\alpha(t))\frac{\mathrm{d}C(t)}{C(t)}+[c-\delta(q(t))]\mathrm{d}t-q(t)\mathrm{d}\sum_{i=1}^{N(t)}Y_i+\sigma_0q(t)\mathrm{d}M_0(t)\\&=[rX^\pi(t)+\alpha(t)\sigma\theta(t)+\lambda(\mu(\eta_0+q(t))-\eta_1\mu_2(1-q(t))^2)]\mathrm{d}t+q(t)\sigma_0\mathrm{d}M_0(t)\\&\quad+\alpha(t)\sigma\mathrm{d}M_1(t)-\int_{R^+}q(t)yN(\mathrm{d}t,\mathrm{d}y)\end{aligned} \tag{5}$$

It was also supposed that the insurance firm possesses a constant absolute risk aversion (CARA) utility $V(x)=-\frac{1}{m}e^{-mx}$, where $m>0$ demonstrates the coefficient of risk aversion. The insurance firm is willing to increase the expected utility of the terminal wealth. So for the traditional ambiguity-neutral insurer (ANI), she has the following objective function:

$$\sup_{\pi\in\Pi}\mathrm{E}\Big[-\frac{1}{m}\exp(-mX^\pi(T))\Big]. \tag{6}$$

\

However, due to the uncertainty of parameters, the insurer only knows some approximate values rather than the real model.

Next, the equivalent set of alternative measures are introduced, i.e., $\mathrm{Q} := \{ ¤ \mid ¤ \sim \mathrm{P} \}$.

**Definition 2.1**

(Admissible strategy) A trading strategy $\pi = \{\alpha(t), q(t)\}_{t\in[0,T]}$ can be accepted if it satisfies

(i) $\forall t \in [0,T]$, $q(t) \in [0,1]$;

(ii) $\pi$ is progressively measurable and $E^{Q^*}[\int_0^T \| \pi\|^4 dt] < \infty$;

(iii) $\forall (t,x) \in [0,T] \times ¡$ , there is a path-wise unique solution of Equation 5, $\{X^{\pi}(t)\}_{t\in[0,T]}$ with $E_{t,x,\theta}^{Q^*}[e^{-mX^{\pi}(t)}] < +\infty$ , where $Q^*$ measures the worst-case.

In the following, we use $\Pi$ to denote the set of whole the admissible strategies.

Consistent with the theorem of Girsanov, we could obtain that for all $¤ \in \mathrm{Q}$ , they are progressively measurable $\phi(t) = (\phi_0(t), \phi_1(t), \phi_2(t), \phi_J(t))^{\bullet}$ , such that $\frac{\mathrm{d}¤}{\mathrm{dP}} = \varphi(T)$ , where

$$\varphi(t) = \exp\{-\int_0^t \bar{\phi}(s)^{\bullet}\, \mathrm{d}M(s) - \frac{1}{2}\int_0^t \| \bar{\phi}(s)\|^2 \mathrm{d}s + \int_0^t \int_{¡^+} \ln\phi_J(s) N(\mathrm{d}s, \mathrm{d}y)$$
$$+ \int_0^t \int_{¡^+} (1-\phi_J(s))\upsilon(\mathrm{d}s, \mathrm{d}y)\} \tag{7}$$

with $M(t) = (M_0(t), M_1(t), M_2(t))^{\bullet}$ and $\bar{\phi}(t) = (\phi_0(t), \phi_1(t), \phi_2(t))^{\bullet}$ . If $\phi(t)$ satisfies Novikov's condition

$\mathrm{E}^{¤}[\exp(\frac{1}{2}\int_0^T \| \phi(s\|^2 ds)] < \infty,$ with $\| \phi(s)\|^2 = \phi_0(t)^2 + \phi_1(t)^2 + \phi_2(t)^2 + \phi_J(t)^2$ , then $\phi(t)$ represents a $\mathrm{P}-$martingale with filtration $\{\mathrm{F}_t\}_{t\in(0,T)}$.

Moreover, the Brownian motions under $¤ \in \mathrm{Q}$ could be explained as

$$\mathrm{d}M^{¤}(t) = \mathrm{d}M(t) + \bar{\phi}(t)\mathrm{d}t, \tag{8}$$

and the intensity of the Poisson procedure $N(t)$ becomes $\phi_J(t)\lambda$. For tractability, similar to Ref. 4, we assume that we know the distribution of the claim $Y_i$, i.e., the distribution $F(y)$ is identical under the measure of $\mathrm{P}$ and $¤$ .

So the surplus procedure under the measure $¤$ could be given as

$$\mathrm{d}X^{\pi}(t) = [rX^{\pi}(t) + \alpha(t)\sigma(\theta(t) - \phi_1(t)) + \lambda(\mu(\eta_0 + q(t)) - \eta_1\mu_2(1-q(t))^2) - \sigma_0 q(t)\phi_0(t)]\mathrm{d}t$$
$$+ q(t)\sigma_0 \mathrm{d}M_0^{¤}(t) + \alpha(t)\sigma \mathrm{d}M_1^{¤}(t) - \int_{R^+} q(t) y N^{¤}(\mathrm{d}t, \mathrm{d}y), \tag{9}$$

with the procedure of the market price of risk as

$$\mathrm{d}\theta(t) = [k(\bar{\theta} - \theta(t)) - \rho\sigma_\theta\phi_1(t) - \sqrt{1-\rho^2}\sigma_\theta\phi_2(t)]\mathrm{d}t + \sigma_\theta[\rho \mathrm{d}M_1^{¤}(t) + \sqrt{1-\rho^2}\mathrm{d}M_2^{¤}(t)]. \tag{10}$$

We suppose that the Ambiguity-Averse Insurer (AAI) faces the reinsurance problem and robust optimal investment under the alternative measure. In other words, the best choice could be found in some worst cases. Following Ref. 4 and Ref. 5, we suppose that the optimization problems below are faced by the insurer.

$$\sup_{\pi\in\Pi}\inf_{¤\in\mathrm{Q}} \mathrm{E}^{¤}\Big[-\frac{1}{m}e^{-mX^{\pi}(T)} + \int_0^T \Big(\frac{R(s)}{\psi(s)} + \frac{(\phi_J(s)\ln\phi_J(s) - \phi_J(s) + 1)\lambda}{\psi_J(s)}\Big)\mathrm{d}s\Big] \tag{11}$$

The expectation is estimated according to the defined different measures $\phi(t)$. The measure is determined by decreasing the expected effect by considering the worst scenario. In addition, deviation from the reference model will also be punished, which

\

is given by the second term of expectation. The weighted relative entropy generated by diffusion risk and jump risk is used for the measurement of this distance. By applying similar methods in Ref. 6, the increment of relative entropy from time $t$ to time $t+\mathrm{d}t$ is equal to

$$[\frac{1}{2}(\phi_0^2(t)+\phi_1^2(t)+\phi_2^2(t))+\lambda(\phi_J(t)\ln\phi_J(t)-\phi_J(t)+1)]\mathrm{d}t \quad (12)$$

.

To deal with the challenge of optimal investment reinsurance for AAI, the value function have been described as follows

$$U(t,x,\theta)=\sup_{\pi\in\Pi}\inf_{\mathbb{Q}\in\mathcal{Q}}\mathrm{E}_{t,x,\theta}^{\mathbb{Q}}\Big[-\frac{1}{m}e^{-mX^{\pi}(T)}+\int_t^T\Big(\frac{R(s)}{\psi(s)}+\frac{(\phi_J(s)\ln\phi_J(s)-\phi_J(s)+1)\lambda}{\psi_J(s)}\Big)ds\Big] \quad (13)$$

with $R(t):=\frac{1}{2}[\phi_0^2(t)+\phi_1^2(t)+\phi_2^2(t)]$ and the boundary condition $U(T,x,\theta)=-\frac{1}{m}e^{-mx}$.

As per the principles of dynamic programming, we have obtained the robust equation of HJB developed in Ref. 6 as follows

$$\sup_{\pi\in\Pi}\inf_{\mathbb{Q}\in\mathcal{Q}}\Big\{\mathcal{A}^{\pi,\mathbb{Q}}U+\frac{\phi_0^2+\phi_1^2+\phi_2^2}{2\psi}+\frac{(\phi_J\ln\phi_J-\phi_J+1)\lambda}{\psi_J}\Big\}=0 \quad (14)$$

Where $\mathcal{A}^{\pi,\mathbb{Q}}$ is the infinitesimal generator under the measure $\mathbb{Q}$, which could be defined as follows

$$\begin{aligned}\mathcal{A}^{\pi,\mathbb{Q}}U=&U_t+[rx+\alpha\sigma(\theta-\phi_1)+\lambda(\mu(\eta_0+q)-\eta_1\mu_2(1-q)^2)-\sigma_0\phi_0 q]U_x\\&+[k(\bar{\theta}-\theta)-\rho\sigma_\theta\phi_1-\sqrt{1-\rho^2}\sigma_\theta\phi_2]U_\theta+\frac{1}{2}(q^2\sigma_0^2+\alpha^2\sigma^2)U_{xx}+\frac{1}{2}\sigma_\theta^2U_{\theta\theta}\\&+\alpha\rho\sigma\sigma_\theta U_{x\theta}+\lambda\phi_J\mathrm{E}^{\mathbb{Q}}[U(t,x-qY,\theta)-U(t,x,\theta)].\end{aligned} \quad (15)$$

Moreover, we suppose that $\psi(t)$ and $\psi_J(t)$ are state-dependent. Following Ref. 5, we set

$$\psi(t)=-\frac{\beta}{mU(t,x,\theta)},$$
$$\psi_J(t)=-\frac{\beta_J}{mU(t,x,\theta)}, \quad (16)$$

where $\beta$ is the coefficient of the ambiguity-aversion for the insurer regarding the uncertainty of the diffusion risk and $\beta_J$ is the ambiguity-aversion coefficient of the insurer for the uncertainty of the jump risk.

## SOLUTIONS

In this section, we focus on the HJB equation (14), and derive the explicit solution.

### Optimal Robust Investment-Reinsurance Strategy

To resolve Eq.(14), a conjecture has been made which holds that the form of the value function $U(t,x,\theta)$ is

$$U(t,x,\theta) = -\frac{1}{m} e^{-mxe^{r(T-t)}+G(t,\theta)}. \tag{17}$$

The partial derivatives of $U(t,x,\theta)$ are as below:

$$U_t = (rmxe^{r(T-t)} + G_t)U, \quad U_x = -me^{r(T-t)}U, \quad U_\theta = G_\theta U,$$

$$U_{xx} = m^2 e^{2r(T-t)}U, U_{\theta\theta} = (G_{\theta\theta} + G_\theta^2)U, \quad U_{x\theta} = -me^{r(T-t)}G_\theta U.$$

Let $f(q) = P_Y(mqe^{r(T-t)}) - 1$, from Eq. (17), we get

$$\mathrm{E}[U(t,x-qY,\theta) - U(t,x,\theta)] = U(t,x,\theta) f(q). \tag{18}$$

The infimum in Eq. (14) will be arrived at as below by using the optimality conditions of the first order

$$\phi_0^*(t) = -\frac{\beta\sigma_0 q U_x}{mU}; \quad \phi_1^*(t) = -\frac{\beta}{m}[\alpha\sigma\frac{U_x}{U} + \rho\sigma_\theta\frac{U_\theta}{U}] \tag{19}$$

$$\phi_2^*(t) = -\frac{\sqrt{1-\rho^2}\beta\sigma_\theta U_\theta}{mV}; \quad \phi_J^*(t) = e^{\frac{\beta_J}{m}[P_Y(mqe^{r(T-t)})-1]} \tag{20}$$

The drift term $\phi_0^*$, $\phi_1^*$, $\phi_2^*$ and $\phi_J^*$ measure the worst-case, and we will derive the optimal strategy under the worst-case. Now substituting (19) and (20) into HJB equation (14), we yield

$$U_t + [rx + \lambda\mu\eta_0]U_x + [k(\bar{\theta} - \theta) + \frac{\beta\sigma_\theta^2 U_\theta}{2mU}]U_\theta + \frac{1}{2}\sigma_\theta^2 U_{\theta\theta} - \frac{m\lambda}{\beta_J}U$$
$$+ \sup_\alpha\{H_1(t,\alpha)\} + \sup_q\{H_2(t,q)\} = 0 \tag{21}$$

Where

$$H_1(t,\alpha) = \frac{1}{2}\Big(U_{xx} + \frac{\beta U_x^2}{mU}\Big)\sigma^2\alpha^2 + \Big(\theta U_x + \rho\sigma_\theta U_{x\theta} + \frac{\beta\rho\sigma_\theta U_\theta U_x}{mU}\Big)\sigma\alpha; \tag{22}$$

$$H_2(t,q) = \frac{1}{2}q^2\sigma_0^2\Big(U_{xx} + \frac{\beta U_x^2}{mU}\Big) + \lambda[\mu q - \eta_1\mu_2(1-q)^2]U_x + \frac{m\lambda}{\beta_J}e^{\frac{\beta_J}{m}f(q)}U. \tag{23}$$

Since $U_{xx} < 0$, $H_1(t,\alpha)$ is a concave function in relation to $\alpha$, its maximizer $\alpha^*(t)$ has the following form

\

$$\alpha^*(t) = -\frac{\theta U_x + \rho\sigma_\theta U_{x\theta} + \beta\rho\sigma_\theta U_\theta U_x/(mU)}{[U_{xx} + \beta U_x^2/(mU)]\sigma} = \Big[\frac{\theta}{(m+\beta)\sigma} + \frac{\rho\sigma_\theta G_\theta}{m\sigma}\Big]e^{-r(T-t)}. \quad (24)$$

Note that

$$\frac{\partial H_2}{\partial q} = me^{r(T-t)}U\Big\{q\sigma_0^2(\beta+m)e^{r(T-t)} - 2\lambda\eta_1\mu_2(1-q) - \lambda\mu + \lambda e^{\frac{\beta_J}{m}f(q)}\mathrm{E}[Ye^{mqe^{r(T-t)}Y}]\Big\}, \quad (25)$$

$$\frac{\partial^2 H_2}{\partial q^2} = me^{2r(T-t)}U\Big\{\sigma_0^2(\beta+m) + 2\lambda\eta_1\mu_2 e^{-r(T-t)} + \lambda\beta_J e^{\frac{\beta_J}{m}f(q)}[\mathrm{E}(Ye^{mqe^{r(T-t)}Y})]^2 + \lambda m e^{\frac{\beta_J}{m}f(q)}\mathrm{E}[Y^2e^{mqe^{r(T-t)}Y}]\Big\} < 0. \quad (26)$$

Therefore, $H_2(t,q)$ is a concave function with respect to q, and its maximizer $\hat{q}(t)$ satisfies

$$\lambda\mu + 2\eta_1\lambda\mu_2(1-q) - q\sigma_0^2(m+\beta)e^{r(T-t)} = \lambda e^{\frac{\beta_J}{m}f(q)}\mathrm{E}[Ye^{mqe^{r(T-t)}Y}]. \quad (27)$$

Similar to the method of Ref. 10, we give the Lemma 3.1.

**Lemma 3.1** For any $t \in [0,T)$, Eq. (27) possesses a unique positive solution $q^*(t) \in (0,1)$.

**Proof.** Let

$$h(q) = \lambda\mu + 2\eta_1\lambda\mu_2(1-q) - q\sigma_0^2(m+\beta)e^{r(T-t)},$$

$$g(q) = \lambda e^{\frac{\beta_J}{m}[P_Y(mqe^{r(T-t)})-1]}\mathrm{E}[Ye^{mqe^{r(T-t)}Y}].$$

Since

$$g'(q) = \lambda e^{r(T-t)+\frac{\beta_J}{m}f(q)}\{\beta_J[\mathrm{E}(Ye^{mqe^{r(T-t)}Y})]^2 + m\mathrm{E}[Y^2e^{mqe^{r(T-t)}Y}]\} > 0;$$

$$g''(q) = \lambda e^{2r(T-t)+\frac{\beta_J}{m}f(q)}[\beta_J^2(\mathrm{E}(Ye^{mqe^{r(T-t)}Y}))^3 + 3m\beta_J\mathrm{E}(Ye^{mqe^{r(T-t)}Y})\mathrm{E}(Y^2e^{mqe^{r(T-t)}Y}) + m^2\mathrm{E}(Y^3e^{mqe^{r(T-t)}Y})] > 0. \quad (28)$$

Thus $g(z)$ is an increasing convex function. $h'(q) = -2\eta_1\lambda\mu_2 - \sigma_0^2(m+\beta)e^{r(T-t)} < 0$, and $h(0) = \lambda\mu + 2\lambda\eta_1\mu_2 > g(0) = \lambda\mu$. In addition, $g(1) > g(0) = \lambda\mu > \lambda\mu - \sigma_0^2(m+\beta)e^{r(T-t)} = h(1)$, so for any fixed $t$ and other parameters, $h(q)$ and $g(q)$ possess a distinct point of intersection at the interval $(0,1)$, which means that there is a unique positive root $q^*(t) \in (0,1)$ that satisfies the equation $h(q) = g(q)$.

By Lemma 3.1, the maximizer $\hat{q}(t) = q^*(t)$. Substituting $q^*(t)$ and (24) into (21) yields the following equation

\

$$G_t + k(\bar{\theta} - \theta)G_\theta + \frac{1}{2}\xi\sigma_\theta^2 G_\theta^2 + \frac{1}{2}\sigma_\theta^2 G_{\theta\theta} - \frac{1}{2m^2\xi}[m\theta + (m+\beta)\rho\sigma_\theta G_\theta]^2 + H(t,q^*) = 0, \tag{29}$$

with the boundary condition $G(T,\theta) = 0$. Where $\xi = 1 + \frac{\beta}{m}$ and

$$H(t,q^*) = -me^{r(T-t)}\lambda\mu\eta_0 + \lambda\frac{m}{\beta_J}(e^{\frac{\beta_J}{m}f(q^*)} - 1) + \frac{1}{2}q^{*2}m^2\sigma_0^2\xi e^{2r(T-t)}$$
$$-\lambda me^{r(T-t)}[\mu q^* - \eta_1\mu_2(1-q^*)^2]. \tag{30}$$

To solve partial differential equation (29), lemma 3.2 is given as follows.

**Lemma 3.2** For the equation $a_1'(t) = \xi_1 a_1^2(t) + \xi_2 a_1(t) + \xi_3$ with the boundary condition $a_1(T) = 0$ and $\Delta := \xi_2^2 - 4\xi_1\xi_3$, its solution has the form as in Table 1.

**Proof.** A similar result can be found in Ref. 10, here we omit it.

By using Lemma 3.2, we present the solution of Eq. (29) as follows.

**Proposition 3.1** With the boundary condition $G(T,\theta) = 0$, the partial differential equation (29) has the solution form as

$$G(t,\theta) = a_1(t)\theta^2 + a_2(t)\theta + a_3(t), \tag{31}$$

where $a_1(t)$ satisfies the Eq. (35), $a_2(t)$ is ascertained by Eq. (36), and $a_3(t)$ is ascertained by Eq. (37).

**Proof.** We assume $G(t,\theta)$ has the form of (31), with $a_i(T) = 0, i = 1,2,3$. Inserting partial derivative $G_t$, $G_\theta$ and $G_{\theta\theta}$ into (29), and separating with $\theta^2$, $\theta$ and other terms, the following three ordinary differential equations are obtained:

$$a_{1'}(t) - \xi_1 a_1^2(t) - \xi_2 a_1(t) - \xi_3 = 0; \tag{32}$$

$$a_{2'}(t) - [k + \rho\sigma_\theta + \xi_1 a_1(t)]a_2(t) + 2k\bar{\theta}a_1(t) = 0; \tag{33}$$

$$a_{3'}(t) - \frac{1}{4}\xi_1 a_2^2(t) + k\bar{\theta}a_2(t) + \sigma_\theta^2 a_1(t) + H(t,q^*) = 0. \tag{34}$$

Where

$$\xi_1 = 2\sigma_\theta^2\xi(\rho^2 - 1), \quad \xi_2 = 2(k + \rho\sigma_\theta), \quad \xi_3 = \frac{1}{2\xi}.$$

Note that $\Delta = 4(k + \rho\sigma_\theta)^2 + 4\sigma_\theta^2(1-\rho^2) > 0$, by Lemma 3.2, the solution of (31) is solved as

$$a_1(t) = \frac{\kappa_1\kappa_2[1 - e^{-\sqrt{\Delta}(T-t)}]}{\kappa_2 - \kappa_1 e^{-\sqrt{\Delta}(T-t)}}, \tag{35}$$

Where

$$\kappa_{1,2} = \frac{-(k+\rho\sigma_\theta) \pm \sqrt{(k+\rho\sigma_\theta)^2 + \sigma_\theta^2(1-\rho^2)}}{2\sigma_\theta^2\xi(\rho^2 - 1)}$$

\

**Table 1**
**The equation solution** $a_1'(t) = \xi_1 a_1^2(t) + \xi_2 a_1(t) + \xi_3$

| $\xi_1$ | $\Delta$ | $A(t)$ | |
|---|---|---|---|
| $\xi_1 = 0$ | | $\frac{\xi_3}{\xi_2}[e^{-\xi_2(T-t)} - 1]$ | *(I)* |
| | $\Delta > 0$ | $\frac{\kappa_1\kappa_2[1-e^{-\sqrt{\Delta}(T-t)}]}{\kappa_2 - \kappa_1 e^{-\sqrt{\Delta}(T-t)}}$, where $\kappa_{1,2} = \frac{-\xi_2 \pm \sqrt{\Delta}}{2\xi_1}$ | *(II)* |
| $\xi_1 \neq 0$ | $\Delta = 0$ | $\frac{-\kappa}{1-\xi_1\kappa(T-t)} + \kappa$, where $\kappa = -\frac{\xi_2}{2\xi_1}$ | *(III)* |
| | $\Delta < 0$ | $\frac{\sqrt{-\Delta}}{2\xi_1}\tan\{\arctan(\frac{\xi_2}{\sqrt{-\Delta}}) - \frac{\sqrt{-\Delta}}{2}(T-t)\} - \frac{\xi_2}{2\xi_1}$ | *(IV)* |

The solution of (32) is

$$a_2(t) = 2k\bar{\theta} e^{-(k+\rho\sigma_\theta)(T-t)-\xi_1\int_t^T a_1(s)ds} \int_t^T a_1(s) e^{(k+\rho\sigma_\theta)(T-s)+\xi_1\int_s^T a_1(u)du} ds. \tag{36}$$

And $a_3(t)$ satisfies

$$\begin{aligned} a_3(t) = & -\frac{\lambda m}{\beta_J}(T-t) + \frac{\lambda m \mu \eta_0}{r}[1-e^{r(T-t)}] - \int_t^T \Big[\frac{1}{4}\xi_1 a_2^2(s) - k\bar{\theta} a_2(s) - \sigma_\theta^2 a_1(s)\Big] ds \\ & + \int_t^T \Big[\frac{\lambda m}{\beta_J} e^{\frac{\beta_J}{m} f(q^*(s))} + \frac{1}{2} m^2 \sigma_0^2 \xi q^{*2}(s) e^{2r(T-s)} - m\lambda(\mu q^*(s) - \eta_1\mu_2(1-q^*(s))^2) e^{r(T-s)}\Big] ds \end{aligned} \tag{37}$$

The above results are concluded in Theorem 3.1 as follows.

**Theorem 3.1** If there is a control policy of $(\pi^*, \phi^*)$ and a function of $U(t,x,\theta)$ that meets the Eq. (14) of robust HJB, then $(\pi^*, \phi^*)$ will become an optimal strategy and $U(t,x,\theta)$ will represent the corresponding value function.

**Proof.** See Appendix.

## Suboptimal Strategies

This section evaluates the importance of considering ambiguity. The loss of wealth equivalent utility suffered by investors due to ignoring the fuzziness of the model is quantified in this section. Firstly, an insurer who ignores ambiguity is considered and the suboptimal strategy is adopted. Secondly, the estimated utility achieved by following the robust optimal strategy and by adopting a suboptimal strategy that ignores the ambiguity is compared. The expected utility function $\tilde{U}(t,x,\theta)$ related to an arbitrary $\pi$ is outlined as

$$\tilde{U}(t,x,\theta) = \inf_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}_{t,x,\theta}\Big[-\frac{1}{m}e^{-mX^{\pi}(T)} + \int_t^T \Big(\frac{R(s)}{\psi(s)} + \frac{(\phi_J(s)\ln\phi_J(s) - \phi_J(s)+1)\lambda}{\psi_J(s)}\Big) ds\Big] \tag{38}$$

\

With the boundary condition $\tilde{U}(t,x,\theta) = -\frac{1}{m}e^{-mx}$, where $R(t) := \frac{1}{2}[\phi_0^2(t) + \phi_1^2(t) + \phi_2^2(t)]$. Note that $\phi_j, j = 0,1,2$, and $\phi_J$, remain endogenously determined and rely on the strategy of investment. Unlike the optimal case, the strategy of reinsurance investment here is pre-defined. And the expected utility is related to $\tilde{\pi}$ satisfies the following minimization problem.

$$\inf_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathcal{A}^{\pi,\mathbb{Q}}\tilde{U} + \frac{\phi_0^2 + \phi_1^2 + \phi_2^2}{2\psi} + \frac{(\phi_J \ln\phi_J - \phi_J + 1)\lambda}{\psi_J}\right\} = 0 \tag{39}$$

where $\mathcal{A}^{\pi,\mathbb{Q}}U$ is defined in (15), the first-order conditions with regard to $\phi_0, \phi_1, \phi_2$ and $\phi_J$ illustrate that the worst-case measure is presented by $\phi_0^*, \phi_1^*, \phi_2^*$ and $\phi_J^*$ in Eq. (39), respectively with $\hat{\alpha}^*$, $\hat{q}^*$ substituting for $\tilde{\alpha}$ and $\tilde{q}$. By substituting these into Eq. (39), we get

$$\tilde{U}_t + [rx + \lambda\mu\eta_0]\tilde{U}_x + [k(\bar{\theta} - \theta) + \frac{\beta\sigma_\theta^2\tilde{U}_\theta}{2m\tilde{U}}]\tilde{U}_\theta + \frac{1}{2}\sigma_\theta^2\tilde{U}_{\theta\theta} - \frac{m\lambda}{\beta_J}\tilde{U} + \tilde{H}_1(t,\hat{\alpha}^*) + \tilde{H}_2(t,\hat{q}^*) = 0 \tag{40}$$

Where

$$\tilde{H}_1(t,\hat{\alpha}^*) = \frac{1}{2}\left(\tilde{U}_{xx} + \frac{\beta\tilde{U}_x^2}{m\tilde{U}}\right)\sigma^2(\hat{\alpha}^*)^2 + \left(\theta\tilde{U}_x + \rho\sigma_\theta\tilde{U}_{x\theta} + \frac{\beta\rho\sigma_\theta\tilde{U}_\theta\tilde{U}_x}{m\tilde{U}}\right)\sigma\hat{\alpha}^*; \tag{41}$$

$$\tilde{H}_2(t,\hat{q}^*) = \frac{1}{2}(\hat{q}^*)^2\sigma_0^2\left(\tilde{U}_{xx} + \frac{\beta\tilde{U}_x^2}{m\tilde{U}}\right) + \lambda[\mu\hat{q}^* - \eta_1\mu_2(1-\hat{q}^*)^2]\tilde{U}_x + \frac{m\lambda}{\beta_J}e^{\frac{\beta_J}{m}f(\hat{q}^*)}\tilde{U}. \tag{42}$$

Assume the solution of Eq. (40) has the form as

$$\tilde{U}(t,x,\theta) = -\frac{1}{m}e^{-mxe^{r(T-t)} + \tilde{a}_1(t)\theta^2 + \tilde{a}_2(t)\theta + \tilde{a}_3(t)}. \tag{43}$$

Plugging the relevant derivatives into Eq. (40), we derive

$$\tilde{a}_1'(t) + 2\xi\sigma_\theta^2\tilde{a}_1^2(t) - [2k + 2\rho\xi\sigma_\theta(1 + 2\rho\sigma_\theta\hat{a}_1(t))]\tilde{a}_1(t) = (1 + 2\rho\sigma_\theta\hat{a}_1(t))[1 - \frac{\xi}{2} - \xi\rho\sigma_\theta\hat{a}_1(t)], \tag{44}$$

$$\begin{aligned}&\tilde{a}_2'(t) + \tilde{a}_2(t)[2\xi\sigma_\theta^2\tilde{a}_1(t) - k - (1 + 2\rho\sigma_\theta\hat{a}_1(t))\rho\sigma_\theta\xi] + 2k\bar{\theta}\tilde{a}_1(t) + \rho\sigma_\theta\hat{a}_2(t)(\xi - 1)\\ &+ 2\rho^2\sigma_\theta^2\xi\hat{a}_2(t)(\hat{a}_1(t) - \tilde{a}_1(t)) = 0,\end{aligned} \tag{45}$$

$$\tilde{a}_3'(t) + \frac{1}{2}\xi\sigma_\theta^2[\tilde{a}_2^2(t) + \rho^2\hat{a}_2(t)(\hat{a}_2(t) - 2\tilde{a}_2(t))] + k\bar{\theta}\tilde{a}_2(t) + \sigma_\theta^2\tilde{a}_1(t) + H(t,\hat{q}^*) = 0. \tag{46}$$

## TWO SPECIAL CASE

### Investment-Only Case

The surplus process

\

$$\begin{aligned} dX^{\pi}(t) &= [rX(t)+\alpha(t)\sigma(\theta(t)-\phi_1(t))+\lambda\mu(\eta_0+1)-\sigma_0\phi_0(t)]dt \\ &+\sigma_0 dM_0^Q(t)+\alpha(t)\sigma dM_1^Q(t)-\int_{R^+} yN^Q(dt,dy) \end{aligned} \tag{47}$$

HJB equation

$$\begin{aligned} &\sup_{\pi\in\Pi}\inf_{\text{¤}\in Q}\{ \\ &\bar{U}_t+[rx+\alpha\sigma(\theta-\phi_1)+\lambda(\mu(\eta_0+1)-\sigma_0\phi_0]\bar{U}_x+[k(\bar{\theta}-\theta)-\rho\sigma_\theta\phi_1-\sqrt{1-\rho^2}\sigma_\theta\phi_2]\bar{U}_\theta \\ &+\frac{1}{2}(\sigma_0^2+\alpha^2\sigma^2)\bar{U}_{xx}+\frac{1}{2}\sigma_\theta^2\bar{U}_{\theta\theta}+\alpha\rho\sigma\sigma_\theta\bar{U}_{x\theta}+\lambda\phi_J \mathrm{E}^{\text{¤}}[\bar{U}(t,x-Y,\theta)-\bar{U}(t,x,\theta)] \\ &+\frac{\phi_0^2+\phi_1^2+\phi_2^2}{2\psi}+\frac{(\phi_J\ln\phi_J-\phi_J+1)\lambda}{\psi_J}\}=0. \end{aligned} \tag{48}$$

We give Proposition 4.1 by following similar derivations of reinsurance-investment case.

**Proposition 4.1** For the problem of investment-only, we get the optimal strategy of investment

$$\bar{\alpha}^*(t)=\Big[\frac{\theta}{(m+\beta)\sigma}+\frac{\rho\sigma_\theta(2a_1(t)\theta+a_2(t))}{m\sigma}\Big]e^{-r(T-t)}, \tag{49}$$

and the optimal value function $\bar{U}(t,x,\theta)$ as follows:

$$\bar{U}(t,x,\theta)=-\frac{1}{m}e^{-mxe^{r(T-t)}+a_1(t)\theta^2+a_2(t)\theta+\bar{a}_3(t)}, \tag{50}$$

Where $a_1(t)$ and $a_2(t)$ are determined by (35) (36), and

$$\begin{aligned} \bar{a}_3(t) &= -\frac{\lambda m}{\beta_J}(T-t)+\frac{\lambda m\mu\eta_0}{r}[1-e^{r(T-t)}]-\int_t^T[\frac{1}{4}\xi_1 a_2^2(s)-k\bar{\theta}a_2(s)-\sigma_\theta^2 a_1(s)]ds \\ &+\frac{\lambda m}{\beta_J}e^{\frac{\beta_J}{m}f(1)}(T-t)+\int_t^T\Big[\frac{1}{2}m^2\sigma_0^2\xi e^{2r(T-s)}-m\lambda\mu e^{r(T-s)}\Big]ds. \end{aligned} \tag{51}$$

## Ambiguity-Neutral Case

In case, the insurers are neutral to the ambiguity which means that coefficients of all ambiguity aversion are $\beta=\beta^J=0$ then the wealth process of ANI under P measure can be described through Equation 5 and objective function can be described through Equation 6 whereas the value function is described as

$$\hat{U}(t,x,\theta)=\sup_{\hat{\pi}\in\hat{\Pi}}\mathrm{E}_{t,x,\theta}[-\frac{1}{m}e^{-mX^{\hat{\pi}}(T)}] \tag{52}$$

where $\hat{\pi}=\{\hat{\alpha}(t),\hat{\pi}(t)\}_{t\in[0,T]}$ . Then the corresponding HJB equation is

$$\begin{aligned} &\hat{U}_t+[rx+\lambda\mu\eta_0]\hat{U}_x+k(\bar{\theta}-\theta)\hat{U}_\theta+\frac{1}{2}\sigma_\theta^2\hat{U}_{\theta\theta}+\sup_\alpha\{\alpha\sigma\theta\hat{U}_x+\frac{1}{2}\alpha^2\sigma^2\hat{U}_{xx}+\alpha\rho\sigma\sigma_\theta\hat{U}_{x\theta}\} \\ &+\sup_q\{\lambda[\mu(\eta_0+q)-\eta_1\mu_2(1-q)^2]\hat{U}_x+\frac{1}{2}q^2\sigma_0^2\hat{U}_{xx}+\lambda f(q)\hat{U}\}=0. \end{aligned} \tag{53}$$

\

After a similar calculation, we give the Proposition 4.2.

**Proposition 4.2**

For those ANI who ignores uncertainty of the model, the optimal strategy of reinsurance-investment becomes $\hat{\pi}^* = \{\hat{\alpha}^*(t), \hat{\pi}^*(t)\}_{t\in[0,T]}$, the optimal $\hat{U}(t,x,\theta)$ are as below:

$$\hat{\alpha}^*(t) = \frac{\theta + \rho\sigma_\theta[2\theta\hat{a}_1(t) + \hat{a}_2(t)]}{m\sigma} e^{-r(T-t)}, \quad (54)$$

and $\hat{q}^*(t)$ meets the given equation:

$$\lambda\mu + 2\eta_1\lambda\mu_2(1-q) - q\sigma_0^2 m e^{r(T-t)} = \lambda \mathrm{E}[Ye^{mqe^{r(T-t)}Y}]. \quad (55)$$

$$\hat{U}(t,x,\theta) = -\frac{1}{m} e^{-mxe^{r(T-t)} + \hat{a}_1(t)\theta^2 + \hat{a}_2(t)\theta + \hat{a}_3(t)}, \quad (56)$$

Where

$$\hat{a}_1(t) = \frac{\hat{\kappa}_1\hat{\kappa}_2[1-e^{-\sqrt{\Delta}(T-t)}]}{\hat{\kappa}_2 - \hat{\kappa}_1 e^{-\sqrt{\Delta}(T-t)}}, \quad \hat{\kappa}_{1,2} = \frac{-(k+\rho\sigma_\theta) \pm \sqrt{(k+\rho\sigma_\theta)^2 + \sigma_\theta^2(1-\rho^2)}}{2\sigma_\theta^2(\rho^2-1)}, \quad (57)$$

$$\hat{a}_2(t) = 2k\bar{\theta} e^{-(k+\rho\sigma_\theta)(T-t) - 2\sigma_\theta^2(\rho^2-1)\int_t^T \hat{a}_1(s)ds} \int_t^T \hat{a}_1(s) e^{(k+\rho\sigma_\theta)(T-s) + 2\sigma_\theta^2(\rho^2-1)\int_s^T \hat{a}_1(u)du} ds, \quad (58)$$

$$\begin{aligned}\hat{a}_3(t) = {} & \frac{\lambda m\mu\eta_0}{r}[1-e^{r(T-t)}] - \int_t^T [\frac{1}{2}\sigma_\theta^2(\rho^2-1)\hat{a}_2^2(s) - k\bar{\theta}\hat{a}_2(s) - \sigma_\theta^2\hat{a}_1(s)]d_s \\ & + \int_t^T \Big[\frac{1}{2}m^2\sigma_0^2\hat{q}^*(s)^2 e^{2r(T-s)} - m\lambda e^{r(T-s)}[\mu\hat{q}^*(s) - \eta_1\mu_2(1-\hat{q}^*(s))^2] + \lambda f(\hat{q}^*(s))\Big]ds.\end{aligned} \quad (59)$$

## CONCLUSION

This research study proposes a robust optimal investment reinsurance strategy for insurance companies. It assumes that the jump-diffusion process is satisfied by the surplus procedure of the insurance company. The insurance companies invest their surplus funds in risk-free assets and a risk asset whose ERR is aligned with the mean-reversion procedure. The price procedure of risk-asset satisfies the stochastic process with a mean reversion rate. We develop the mathematical model by increasing the exponential utility of the estimated wealth at the terminal time. Considering the uncertainty of the model, the ambiguity-averse insurance company aims to seek a robust reinsurance and asset allocation strategy. By using the robust stochastic control method, we derivative the robust HJB equation which is satisfied by the value function. And then by guessing the form of the value function, we obtain three ordinary differential equations. Besides, two special cases, investment-only cases, and ambiguity-neutral cases are also explored.

## APPENDIX

According to Corollary 1.2 of Ref. 11, if $\pi^*$ and $V(t, X^{\pi^*}(t), \theta(t))$ fulfills the enlisted properties, Theorem 3.1 could be easily proved.

Property (1)

$\pi^*$ is an admissible strategy.

Property (2)

$\mathrm{E}^{¤^*}[\sup_{t\in[0,T]} \left|U(t, X^{\pi^*}(t), \theta(t))\right|^4] < \infty$, where ¤ $^*$ is defined by $\varphi^*(t)$ with $\phi^*(t)$;

Property (3)

$\mathrm{E}^{¤^*}\Big[\sup_{t\in[0,T]} \left|\frac{R^*(t)}{\psi(t)} + \frac{\lambda(\phi_J^*(t)\ln\phi_J^*(t) - \phi_J^*(t)+1)}{\psi_J(t)}\right|^2\Big] < \infty$, with $R^*(t) = \frac{1}{2}[\phi_0^*(t)^2 + \phi_1^*(t)^2 + \phi_2^*(t)^2]$.

In the following, we prove the above properties respectively.

\

**Proof of Property (1)**

To prove property (1), we could verify the conditions defined in Definition 2.1. Firstly, condition (i) could be verified according to Lemma 3.1. Furthermore, we could also know $\pi^*$ is deterministic and state-independent from Lemma 3.1, which means the condition (ii) is valid. Besides, the validation of condition (iii) could be derived from the given property (2).

**Proof of Property (2)**

By substituting Eq. (24) and $q^*(t)$ which satisfies Eq. (27) into the wealth procedure (9), the optimal wealth procedure is obtained as follows

$$\begin{aligned} X^{\pi^*}(t) = \quad & x_0 e^{rt} + \frac{m}{m+\beta} e^{-r(T-t)} \int_0^t [A_1(s)\theta(s)^2 + A_2(s)\theta(s)]\mathrm{d}s + e^{rt}\int_0^t A_3(s)\mathrm{d}s \\ & + e^{rt}\int_0^t \sigma_0 e^{-rs} q^*(s)\mathrm{d}M_0^{\text{¤}^*}(s) + e^{-r(T-t)}\int_0^t [A_1(s)\theta(s) + A_2(s)]\mathrm{d}M_1^{\text{¤}^*}(s) \\ & - e^{rt}\int_0^t\int_{R^+} e^{-rs} q^*(s) y N^{\text{¤}^*}(\mathrm{d}s, \mathrm{d}y), \end{aligned} \tag{60}$$

Where

$$A_1(s) = \frac{1}{m+\beta} + \frac{2\rho\sigma_\theta a_1(s)}{m}, \quad A_2(s) = \frac{\rho\sigma_\theta a_2(s)}{m},$$

$$A_3(s) = \lambda[\mu(\eta_0 + q^*(s)) - \eta_1\mu_2(1-q^*(s))^2]e^{-rs} - \beta\sigma_0^2(q^*(s))^2 e^{r(T-2s)}. \tag{61}$$

In the following, we write $A_i$ as $A_i(s), i = 1,2,3$ for short. Inserting Eq. (60) into candidate value function, for appreciate constant $K_1 > 0$, we have the following estimate

$$\begin{aligned} & |U(t, X^{\pi^*}(t), \theta(t))^4| \\ & = \left| \frac{1}{m^4} e^{-4mX^{\pi^*}(t)e^{r(T-t)} + 4a_1(t)\theta^2 + 4a_2(t)\theta + 4a_3(t)} \right| \le K_1 e^{-4me^{r(T-t)}X^{\pi^*}(t)} \\ & = K_1 e^{-4me^{rT}[x_0 + \int_0^t A_3\mathrm{d}s - \int_0^t\int_{R^+} e^{-rs}q^*(s)yN^{\text{¤}^*}(\mathrm{d}s,\mathrm{d}y)]} \\ & \quad \cdot e^{-4m\left[\frac{m}{m+\beta}\int_0^t (A_1\theta^2 + A_2\theta)\mathrm{d}s + e^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0\mathrm{d}M_0^{\text{¤}^*}(s) + \int_0^t (A_1\theta + A_2)dM_1^{\text{¤}^*}(s)\right]} \\ & \le K_2 e^{-4m\left[\frac{m}{m+\beta}\int_0^t (A_1\theta^2 + A_2\theta)ds + e^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0\mathrm{d}M_0^{\text{¤}^*}(s) + \int_0^t (A_1\theta + A_2)dM_1^{\text{¤}^*}(s)\right]} \end{aligned} \tag{62}$$

Where $K_2$ is a constant, and it satisfies $K_2 \ge K_1 e^{-4me^{rT}[x_0 + \int_0^t A_3 ds - \int_0^t\int_{R^+} e^{-rs}q^*(s)yN^{\text{¤}^*}(ds,dy)]}$. Since $a_1, a_2$ and $a_3$ are deterministic, which bounded on $[0,T]$, then the first inequality holds. The second estimate holds because of $x_0$, $\int_0^t A_3 ds$ and $\int_0^t\int_{R^+} e^{-rs}q^*(s)yN^{\text{¤}^*}(ds,dy)$ is deterministic and bounded.

In the next step, it was initially considered the integral $e^{-4me^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0 dM_0^{\text{¤}^*}(s)}$. Note that

$$e^{\int_0^t -4me^{r(T-s)}q^*(s)\sigma_0 dM_0^{\text{¤}^*}(s)} = \underbrace{e^{-\int_0^t 8m^2e^{2r(T-s)}(q^*(s))^2\sigma_0^2 ds - \int_0^t 4me^{r(T-s)}q^*(s)\sigma_0 dM_0^{\text{¤}^*}(s)}}_{martingale} \cdot \underbrace{e^{\int_0^t 8m^2e^{2r(T-s)}(q^*(s))^2\sigma_0^2 ds}}_{const.},$$

thus, we obtain

$$\mathrm{E}^{\text{¤}^*}[e^{-4me^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0 dM_0^{\text{¤}^*}(s)}] < \infty. \tag{63}$$

\

Next, we consider the integral $e^{-4m[\frac{m}{m+\beta}\int_0^t (A_1\theta^2+A_2\theta)ds+\int_0^t (A_1\theta+A_2)dM_1^{¤^*}(s)]}$, and rewrite it as:

$$e^{-4m[\frac{m}{m+\beta}\int_0^t (A_1\theta^2+A_2\theta)ds+\int_0^t (A_1\theta+A_2)dM_1^{¤^*}(s)]} = \underbrace{e^{-\int_0^t 16m^2(A_1\theta+A_2)^2ds-\int_0^t 4m(A_1\theta+A_2)dM_1^{¤^*}(s)}}_{F}$$

$$\cdot \underbrace{e^{\int_0^t 16m^2(A_1\theta+A_2)^2ds-\int_0^t \frac{4m^2}{m+\beta}(A_1\theta^2+A_2\theta)ds}}_{G} \cdot$$

For term $F$, we could derive the following mathematical forms:

$$\mathrm{E}^{¤}(F^2)=\mathrm{E}^{¤^*}[e^{-\int_0^t 32m^2(A_1\theta+A_2)^2ds-\int_0^t 8m(A_1\theta+A_2)dM_1^{¤^*}(s)}]<\infty. \tag{64}$$

It is a super-martingale for $F^2$ is a nonnegative local martingale.

For term $G$, the below expressions is put forward:

$$G^2=e^{\int_0^t 32m^2A_2^2ds}\cdot\underbrace{e^{\int_0^t 8m^2A_1(4A_1-\frac{1}{m+\beta})\theta^2ds}}_{G_1}\cdot\underbrace{e^{\int_0^t 8m^2A_2(8A_1-\frac{1}{m+\beta})\theta ds}}_{G_2}\cdot$$

Note that $e^{\int_0^t 32m^2A_2^2ds}$ is deterministic and bounded. Thus, with a constant $K_3\geq e^{\int_0^t 32m^2A_2^2ds}$,

$$\mathrm{E}^{¤^*}[G^2]\leq K_3\mathrm{E}^{¤^*}[G_1G_2]. \tag{65}$$

Using the technical assumption:

$$\mathrm{E}^{¤^*}[G_1^2]=\mathrm{E}^{¤^*}[e^{\int_0^t 16m^2A_1(4A_1-\frac{1}{m+\beta})\theta^2ds}]<\infty,$$

$$\mathrm{E}^{¤^*}[G_2^2]=\mathrm{E}^{¤^*}[e^{\int_0^t 16m^2A_2(8A_1-\frac{1}{m+\beta})\theta ds}]<\infty,$$

and applying (63)-(65), we reach at the following expressions:

$$\begin{aligned}
&\mathrm{E}^{¤^*}|U(t,X^{\pi^*}(t),\theta(t))^4|\\
&\leq K_2\mathrm{E}^{¤^*}[e^{-4me^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0 dM_0^{¤^*}(s)}]\\
&\leq K_2\mathrm{E}^{¤^*}[e^{-4me^{rT}\int_0^t e^{-rs}q^*(s)\sigma_0 dM_0^{¤^*}(s)}]\cdot\mathrm{E}^{¤^*}[e^{-4m[\frac{m}{m+\beta}\int_0^t (A_1\theta^2+A_2\theta)ds+\int_0^t (A_1\theta+A_2)dM_1^{¤^*}(s)]}]\\
&\leq K_4\mathrm{E}^{¤^*}(GF)\leq K_4[\mathrm{E}^{¤^*}(G^2)\mathrm{E}^{¤^*}(F^2)]^{1/2}\\
&\leq K_5[\mathrm{E}^{¤^*}(G_1G_2)]^{1/2}[\mathrm{E}^{¤^*}(F^2)]^{1/2}\\
&\leq K_6[\mathrm{E}^{¤^*}(G_1^2)\mathrm{E}^{¤^*}(G_2^2)]^{1/4}<\infty
\end{aligned}$$

The expectation in the right hand of the first inequity can be dispersed into two parts since $M_0^{¤^*}(t)$ is independent $M_1^{¤^*}(t)$. According to Eq. (63), the second estimate is valid. The third, fourth, and fifth estimates follow from Cauchy-Schwarz inequality.

**Proof of Property (3)**

Suppose $\Phi(t)=\dfrac{mR^*(t)}{\beta}+\dfrac{\lambda m(\phi_J^*(t)\ln\phi_J^*(t)-\phi_J^*(t)+1)}{\beta_J}$, which is bounded function. According to (16), we obtain

\

$$
\begin{aligned}
&\mathrm{E}^{\text{¤}^*}\Big[\sup_{t\in[0,T]}\Big|\frac{R^*(t)}{\psi(t)}+\frac{\lambda(\phi_J^*(t)\ln\phi_J^*(t)-\phi_J^*(t)+1)}{\psi_J(t)}\Big|^2\Big] \\
&=\mathrm{E}^{\text{¤}^*}\Big[\sup_{t\in[0,T]}\big|\Phi(t)\big|^2\big|U(t,X^{\pi^*}(t),\theta(t))\big|^2\Big] \qquad (66) \\
&\leq\Big(\mathrm{E}^{\text{¤}^*}\Big[\sup_{t\in[0,T]}\big|\Phi(t)\big|^4\Big]\Big)^{1/2}\Big(\mathrm{E}^{\text{¤}^*}\Big[\sup_{t\in[0,T]}\big|U(t,X^{\pi^*}(t),\theta(t))\big|^4\Big]\Big)^{1/2}<\infty
\end{aligned}
$$

Thus property (3) is proven.

Based on the properties (1)-(3), we could use Corollary 1.2 in Ref. 11 to prove Theorem 3.1.

## Author Declaration

This research is not funded by any organization related to tobacco production.

## Reference

1. Hata H, Sheu SJ. An optimal consumption and investment problem with partial information. *Advances in Applied Probability*. 2017; 50(1):131-153.

2. Bai L, Zhang H. Dynamic mean-variance problem with constrained risk control for the insurers. *Mathematical Methods of Operations Research.* 2008; 68(1): 181-205.

3. Zeng Y, Li Z. Optimal time-consistent investment and reinsurance policies for mean-variance insurers. *Insurance: Mathematics and Economics*. 2011; 49 (1): 145-154.

4. Branger N, Larsen LS. Robust portfolio choice with uncertainty about jump and diffusion risk. *Journal of Banking and Finance.* 2013; 37(12), 5036-5047.

5. Maenhout, PJ. Robust portfolio rules and asset pricing. *Review of Financial Studies*. 2004; 17(4): 951-983.

6. Maenhout PJ. Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium. J*ournal of Economic Theory*. 2006; 128(1): 136-163.

7. Gu AL, Viens FG, Yao HX. Optimal robust reinsurance-investment strategies for insurers with mean reversion and mispricing. *Insurance: Mathematics and Economics.* 2018; 80(5): 93-109.

8. Guan GH, Liang ZX. Robust optimal reinsurance and investment strategies for an AAI with multiple risks. *Insurance: Mathematics and Economics*. 2019; 89(11): 63-78.

9. Zhang Q, Chen P. Robust optimal proportional reinsurance and investment strategy for an insurer with defaultable risks and jumps. *Journal of Computational and Applied Mathematics.* 2019; 356(4): 46-66.

10. Lin X, Yang P. Optimal investment and reinsurance in a jump diffusion risk model. *The Anziam Journal*. 2011; 52(3): 250-262.

11. Shen Y, Zeng Y. Optimal investment–reinsurance with delay for mean–variance insurers: A maximum principle approach. *Insurance: Mathematics and Economics*. 2014; 57: 1-12.

12. Li D, Rong X, Zhao H. Optimal investment problem for an insurer and a reinsurer. *Journal of Systems Science and Complexity.* 2015; 28(6): 1326-1343.

13. Sun Z, Zheng X, Zhang X. Robust optimal investment and reinsurance of an insurer under variance premium principle and default risk. *Journal of Mathematical Analysis and Applications.* 2017; 446(2): 1666-1686.

14. Zhang X, Meng H, Xiong J, et al. Robust optimal investment and reinsurance of an insurer under Jump-diffusion models. *Mathematical Control & Related Fields.* 2019; 9(1): 59.

15. Sun J, Yao H, Kang Z. Robust optimal investment–reinsurance strategies for an insurer with multiple dependent risks. *Insurance: Mathematics and Economics*. 2019; 89: 157-170.

16. Chen Z, Yang P. Robust optimal reinsurance–investment strategy with price jumps and correlated claims, Insurance: *Mathematics and Economics*. 2020; 92: 27-46.

17. Li Q, Gu M. Optimal reinsurance and investment policies with the CEV stock market, Acta Mathematicae Applicatae Sinica, English Series. 2016; 32(3): 647-658.

18. Huang Y, Yang X, Zhou J. Optimal investment and proportional reinsurance for a jump–diffusion risk model with constrained control variables. *Journal of Computational and Applied Mathematics*. 2016; 296: 443-461.

19. Wang P, Li Z. Robust optimal investment strategy for an AAM of DC pension plans with stochastic interest rate and stochastic volatility. *Insurance: Mathematics and Economics*. 2018; 80: 67-83.

20. Zhao H, Shen Y, Zeng Y, et al. Robust equilibrium excess-of-loss reinsurance and CDS investment strategies for a mean–variance insurer with ambiguity aversion. *Insurance: Mathematics and Economics*. 2019; 88: 159-180.

21. Wang N, Zhang N, Jin Z, et al. Robust non-zero-sum investment and reinsurance game with default risk. *Insurance: Mathematics and Economics.* 2019, 84: 115-132.

22. Liu B, Meng H, Zhou M. Optimal investment and reinsurance policies for an insurer with ambiguity aversion. *The North American Journal of Economics and Finance.* 2021; 55: 101303.

23. Liang, Z., Yuen, K. C. Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scandinavian Actuarial Journal*. 2016; 2016 (1): 8-36.

24. Kraft H. Optimal portfolios with stochastic interest rates and defaultable assets. 2004; Berlin: Springer-Verlag

\